

SHORT COMMUNICATIONS

Numerical tests of efficiency of the retrospective time integration scheme in the self-memory model^{*}

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Abstract A set of numerical tests was carried out to compare the retrospective time integral scheme in a self-memory model, whose dynamic kernel is the barotropical quasi-geostrophic model, with the ordinary centered difference scheme in the barotropical quasi-geostrophic model. The Rossby-Haurwitz wave function was taken as the initial fields for both schemes. The results show that in comparison with the ordinary centered difference scheme, the retrospective time integral scheme reduces by 2 orders of magnitude the forecast error, and the forecast error increases very little with lengthening of the time-step. Therefore, the retrospective time integral scheme has advantages of improving the forecast accuracy, extending the predictable duration and reducing the computation amount.

Keywords: numerical tests self-memory model retrospective time integral scheme.

Although the design of computing scheme in the numerical weather prediction is mainly to improve the spacial difference, the research for improving the time difference scheme has seen apparent progresses^[1~5]. It is confirmed that the retrospective time integral scheme adopted in a spectrum model, which utilizes multi-time historical data, improves the accuracy of the numerical weather prediction^[6,7]. In this article, we apply the retrospective time integral scheme to a simple difference model, perform numerical experiments and present a comparison analysis.

1 Principle and formula

For convenience, the simplest barotropical quasi-geostrophic model, i. e. the barotropical vorticity equation model

$$\frac{\partial q}{\partial t} = J(q + f, \psi) \quad (1)$$

was employed as a dynamic kernel of the self-memory model. Here q denotes the relative vorticity, ψ is the stream function, $J(\cdot)$ represents the Jacobin operator, and other symbols are all commonly used in meteorology. By changing the time integration into a retrospective time integral scheme, which uses multi-time historical data, a corresponding difference self-

memory model was derived as^[8]

$$q_t = \alpha_{-p} q_{-p} + \sum_{j=-p}^0 \theta_j q_j^m - \sum_{j=-p}^0 r_j J(\psi, q_j) \Delta t - \sum_{j=-p}^0 \epsilon_j J(\psi, f) \Delta t \quad (2)$$

where α , θ , γ , ϵ are coefficients related to the self-memory function, and p is a retrospective order. Let

$$y_j = q_j^m = \frac{1}{2} [q(t_j) + q(t_{j+1})], \quad q_{-p} = q_{-p-1}^m.$$

Then (1) may be rewritten as

$$q_t = \sum_{j=-p-1}^0 \theta_j y_j - \sum_{j=-p}^0 r_j J(\psi, q_j) - \sum_{j=-p}^0 \epsilon_j J(\psi, f). \quad (3)$$

For the historical data^[7] including L -times before the time $-p$, let

$$\mathbf{X}_t = \begin{bmatrix} q_{t1} \\ \vdots \\ q_{tL} \end{bmatrix}, \quad \mathbf{\Theta} = \begin{bmatrix} \theta_{-p} \\ \vdots \\ \theta_0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} r_{-p} \\ \vdots \\ r_0 \end{bmatrix},$$

$$\mathbf{E} = \begin{bmatrix} \epsilon_{-p} \\ \vdots \\ \epsilon_0 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} y_{-p-1,1} & \cdots & y_{-1,1} \\ \cdots & \cdots & \cdots \\ y_{-p-1,L} & \cdots & y_{-1,L} \end{bmatrix},$$

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$$\mathbf{P} = \begin{bmatrix} J_1(\psi_p, q_p) \cdots J_1(\psi_0, q_0) \\ \cdots \cdots \\ J_L(\psi_p, q_p) \cdots J_L(\psi_0, q_0) \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} J_1(\psi_p, f) \cdots J_1(\psi_0, f) \\ \cdots \cdots \\ J_L(\psi_p, f) \cdots J_L(\psi_0, f) \end{bmatrix}.$$

Then (3) can be rewritten as the prediction equation in a matrix form:

$$\mathbf{X}_t = \mathbf{Y}\Theta + \mathbf{P}\mathbf{R} + \mathbf{F}\mathbf{E} = \mathbf{Z}\mathbf{W}, \quad (4)$$

where

$$\mathbf{Z} = [\mathbf{Y} \quad \mathbf{P} \quad \mathbf{F}], \quad \mathbf{W} = \begin{bmatrix} \Theta \\ \mathbf{R} \\ \mathbf{E} \end{bmatrix}. \quad (5)$$

To calculate the matrix \mathbf{W} with multi-time historical data, we obtained the least-square solution of the matrix \mathbf{W} from (4) as the following, based on the generalized least-square method.

$$\mathbf{W} = (\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{Z}^T \mathbf{X}_t, \quad (6)$$

where $()^T$ represents the transpose of the matrix, $()^{-1}$ denotes the matrix inversion. Thus, we may firstly calculate the coefficient matrix \mathbf{W} in the self-memory model equation from (6) utilizing the historical data in L -times before the time $-p$, then compute the predictand at time $t=1$ from (3) with the previous data from time $-p-1$ to time 0 (usually $L \geq 30$). The Haurwitz wave^[9] on the wave-number equal to 4 was taken as the initial field^[8] at $\Delta\lambda = \Delta\varphi = 5^\circ$ for the globe ($0^\circ\text{E} \sim 360^\circ$, $90^\circ\text{S} \sim 360^\circ\text{N}$) in the integral computation.

For comparison, the dynamic kernel, the barotropical quasi-geostrophic model, was also integrated with the same initial field, space grid length $\Delta\lambda = \Delta\varphi = 5^\circ$ and time step length Δt for the globe, except that an ordinary centered difference scheme (forward difference at the first step) was taken instead of the retrospective time integral scheme.

2 Comparison of prediction results

2.1 The retrospective scheme gives much greater predictive accuracy

The retrospective time integral scheme in the self-memory model was integrated at one time-step (see section 3 about the reason) of different lengths $\Delta t = 1\text{ h}, 2\text{ h}, \dots, 2160\text{ h}$ (90 days). Meanwhile, the ordinary centered difference time integral scheme in the barotropical quasi-geostrophic model was also integrated at time-step length $\Delta t = 1\text{ h}$ for different durations up to 2160 h (90 days). This follows the

C.F.L. difference computation stable condition, because the barotropical quasi-geostrophic model is a filtered model, in which $\Delta t = 1\text{ h}$ is usually used in prediction tests. The predictive value of the relative vorticity and its root-mean-square error (RMSE) from the precise value (i.e. the Rossby-Haurwitz wave analytical solution) were separately calculated with both the above mentioned schemes.

Fig. 1(a) shows the results integrated for 5 days from the two schemes. In the ordinary difference scheme the predictive error increases obviously with lengthening of the integral time during the first 24 hours of prediction. The order of the predictive error reaches up to the same order as the precise solution (10^{-5}) at and after the 24th hour. The root-mean-square of the predictive error (3.75×10^{-5}) exceeds the precise solution itself (3.13×10^{-5}) at the 48th hour of the prediction, i.e. the relative error is more than 100%, which is not practicable.

However, it is inspiring that the retrospective scheme produces much less predictive error at the order of magnitude $10^{-8} \sim 10^{-7}$ and keeps stable till the 90th day of the prediction. The predictive error order is 2~3 orders less than that from the ordinary difference scheme, which is 10^{-5} .

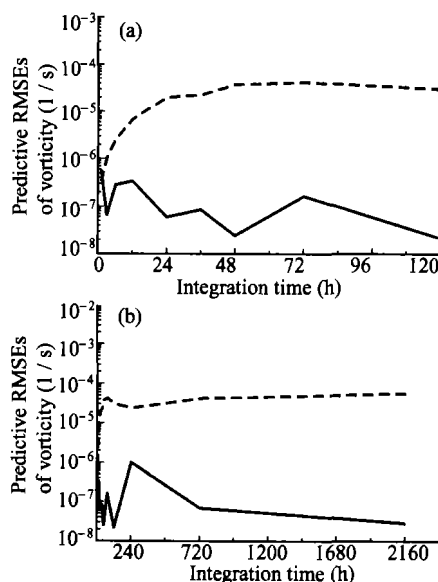


Fig. 1. The predictive RMSEs of vorticity from the retrospective scheme (—) and the ordinary difference scheme (---), (unit: s^{-1}). (a) For 5-day integration; (b) for 90-day integration.

2.2 The retrospective scheme is insensible to the integral time-step length Δt

The retrospective time scheme not only has the

advantage of greater predictive accuracy, but also appears specially insensible to the integral time-step length Δt . Fig. 2 presents predictive root-mean-square errors (RMSEs) of vorticity calculated from the two schemes with only one-step integration at various time-step lengths $\Delta t = 1$ h, 2 h, ..., 2160 h (90 days) respectively. These numerical tests are just for comparison, although the ordinary difference scheme violates the C.F.L. condition for long time-step length.

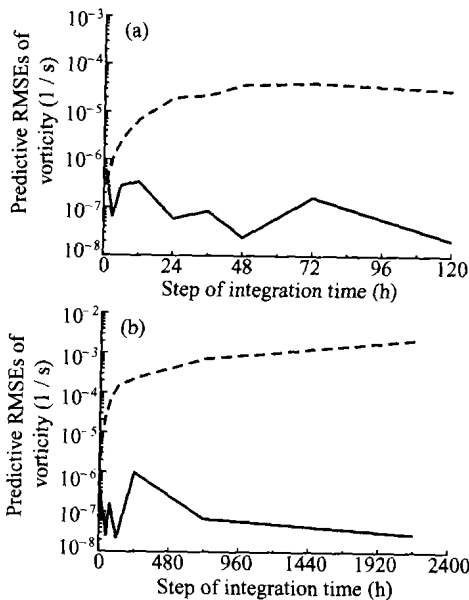


Fig. 2. The predictive RMSEs of vorticity from the retrospective scheme (—) and from the ordinary difference scheme (---) with one-step integration at various time-step lengths Δt (unit: s^{-1}). (a) For Δt lengthening up to 5 days; (b) for Δt lengthening up to 90 days.

Figure 2 shows very interesting results. The predictive error from the ordinary difference scheme increases monotonously with lengthening of Δt for $\Delta t > 48$ h, though the RMSE increases slower than that for $\Delta t < 48$ h. The order of magnitude of the RMSE reaches up to the same as the precise solution itself (10^{-5}), when the ordinary difference scheme is integrated with one-step at step-length $\Delta t = 24$ h. The RMSE reaches up to 3.24×10^{-5} , which exceeds the precise solution (3.13×10^{-5}) in one-step integration at step-length $\Delta t = 48$ h. In contrast, the RMSE order from the retrospective scheme keeps so low as $10^{-8} \sim 10^{-7}$ at all times with little increase as Δt lengthens, i.e. it is insensible to the integral time-step length. Therefore, if there are long enough and precise historical data, the Δt may be much lengthened by applying the retrospective time integral scheme, so that the computing amount may be de-

creased and the predictable duration in practice may be increased.

In addition, the ordinary centered difference scheme was integrated continuously with multi-steps at step-lengths $\Delta t = 1, 3, 6$ h respectively. The predictive RMSEs of vorticity (see Fig. 3) increase rapidly up to 10^0 or more at $\Delta t = 3$ and 6 h, and then the computing overflow happened at the 48th or 72nd hour respectively. This confirms the basic principle of the difference method, i.e. the time-step length Δt is restrained with the space-grid length Δs in $\Delta t \leq |\Delta s / \sqrt{2}c|$ (c is wave speed), which is the so-called C.F.L. condition. Therefore, the time-step length Δt cannot be taken as too long, and consequently, the computation amount and the predictable duration are all limited in practice.

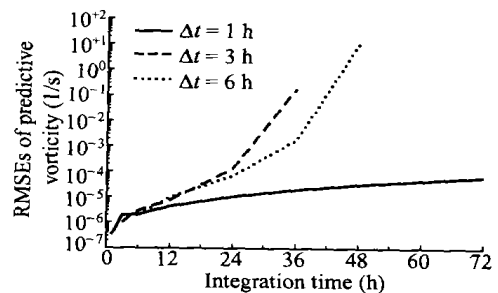


Fig. 3. The RMSEs of predictive vorticity from the ordinary centered difference scheme (unit: s^{-1}).

3 Choice of the dynamic kernel

To choose a dynamic kernel of the retrospective time integral scheme, we need to consider not only its features, such as stability, convergence, forecast accuracy, etc., but also a speciality for computation of the retrospective scheme.

The barotropic quasi-geostrophic model with the ordinary difference scheme may be written as $^2(\partial\psi/\partial t) = J(q + f, \psi)$, where ψ denotes the stream function, and q is the relative vorticity. The Poisson Equation is firstly iterated to obtain the $\partial\psi/\partial t$, which is usually relatively small in the order of magnitude of 10^4 . Then by integrating $\psi^{(n+1)} = \psi^{(n)} + \Delta t (\partial\psi/\partial t)^{(n)}$ in time with the centered difference scheme, we get the stream function $\psi^{(n+1)}$ for the next step-time, which is comparatively accurate and stable. For $\Delta t \leq 3$ h, the order of magnitude of predictive error of the stream function within 15 hours of integration is about 2 orders less than the precise solution (the Rossby-Haurwitz wave analytic

solution, 3.13×10^{-5}).

In the retrospective scheme, however, the dynamic kernel—the barotropical quasi-geostrophic model—is written as $\partial q / \partial t = J(q + \psi, \psi)$, and the time integration has to be solved first so that the multi-time historical data may be used to determine the $q^{(n+1)}$ according to Eq. (3). Then the Poisson Equation $\nabla^2 \psi^{(n+1)} = q^{(n+1)}$ is solved iteratively to predict the $\psi^{(n+1)}$. In this process of solving the Poisson Equation, the order of magnitude of the $q^{(n+1)}$ is about 10^{-5} , which is corresponding to 10^9 of the $\psi^{(n+1)}$. The difference of the orders of magnitude between the $q^{(n+1)}$ and $\psi^{(n+1)}$ is about 10^{14} . This may lead to a large error of the predictive $\psi^{(n+1)}$ due to iterating computation to solve the Poisson Equation. The actual $\psi^{(n+1)}$ is on the order of 10^7 in our practical computation, i.e. the order of error of the predictive $\psi^{(n+1)}$ is the same as that of the ψ itself. If this $\psi^{(n+1)}$ were continuously taken into the time integration, the $\psi^{(n+2)}$ would have a larger error, and would lose the predictability.

Therefore, we compute the time integration with only one-step to determine the vorticity $q^{(n+1)}$ in the retrospective scheme. Fortunately, this predictive accuracy is apparently greater than that from the ordinary difference scheme. Furthermore, the prediction is insensitive to the time-step length. Thus we may take a longer step-length Δt in the retrospective scheme. A successful annual forecast experiment was reported with one-step integration in a regional climatic self-memory model^[10], which produced a considerably great accuracy and good effect.

In addition, we failed to improve the iterating solution of the Poisson Equation by adopting a double precision computation, or different initial values and boundary conditions, even heightening the calculating precision request, etc. The reason might be the limitation of the iterating solution itself in the Poisson equation.

There might be 3 ways to overcome this difficulty: (1) to search for a better scheme to solve the Poisson Equation; (2) to select another model as the dynamic kernel that is independent of the sequence of

time integral, such as the shallow water wave equations, or baroclinic primitive equation model etc.; and (3) to lengthen the time-step length based on the advantage of insensibility to time-step length and of higher accuracy in the retrospective scheme.

4 Concluding remarks

The retrospective time integral scheme was adopted in a difference model in this article. The numerical tests show that, compared to the ordinary difference scheme, the retrospective time integral scheme produces a reduction of 2 orders of magnitude of the predictive error based on an ideal initial field, and the time integration is insensitive to the step-length. This result is inspiring.

However, it would bring a large predictive error of the stream function if multi-step (more than one-step) integration were computed in the barotropical quasi-geostrophic model as the dynamic kernel, because the time-integration must be firstly performed. Therefore, it would be better to choose other dynamic kernels in the future studies.

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